The Non-Split Scalar Coset in Supergravity Theories

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Abstract

The general non-split scalar coset of supergravity theories is discussed.The symmetric space sigma model is studied in two equivalent formulations and for different coset parametrizations.The dualisation and the local first order formulation is performed for the non-split scalar coset G/K when the rigid symmetry group G is a real form of a non-compact semisimple Lie group (not necessarily split) and the local symmetry group K is G's maximal compact subgroup.A comparison with the scalar cosets arising in the T^{10-D}-compactification of the heterotic string theory in ten dimensions is also mentioned.

1 Introduction

The first order formulation of the maximal supergravities ($D \leq 11$) has been given in [1,2] where the scalar sectors which are governed by the G/K symmetric space sigma model are studied case by case.Dualisation of the fields and the generators which parametrize the coset is the method used to obtain locally the first order equations as a twisted self duality condition in [1,2].In [3] a general formulation based on the structure constants is given to construct an abstract method of dualisation and to derive the first order

equations for the G/K coset sigma model when the rigid symmetry group G is in split real form (maximally non-compact).Although the maximal supergravities fall into this class there are more general cases of pure or matter coupled supergarvity theories whose scalar sectors posses a non-split rigid symmetry group.When the Bosonic sector of the ten dimensional simple supergravity coupled to N Abelian gauge multiplets is compactified on T^{10-D} the scalar sector of the D-dimensional reduced theory can be formulated as G/K symmetric space sigma model with G being in general non-split (not maximally non-compact) [4].In particular when N is chosen to be 16 the formulation and the coset realizations of [4] correspond to the D-dimensional Kaluza-Klein reduction of the low-energy effective heterotic string theory in ten dimensions.

In this work by presenting an algebraic outline we are enlarging the first order formulation of [3] by using the solvable algebra parametrization [5,6] to a more general case which contains a rigid symmetry group G which is a real form of a non-compact semisimple Lie group and G is not necessarily in split real form.We will choose a different spacetime signature convention than the one assumed in [3].We will assume $s = 1$ whereas in [3] it has been taken as $s = (D-1)$.Therefore there will be a sign factor depending on the spacetime dimension in the first-order equations.The formalism presented here covers the split symmetry group case as a particular limit in the possible choice of the non-compact rigid symmetry groups as it will be clear in section two.Before giving the first order formulation we will present two equivalent definitions of the G/K scalar coset sigma model and we will study them in detail.We will introduce two coset maps,one being the parametrization used in [1,2,3].A transformation law between the two sets of scalar fields will also be established to be used to relate the corresponding first order equations.The second order equations of the vielbein formulation and the internal metric formulation will be derived for both of these coset maps.We will also discuss the correspondence of our formulation with the scalar manifold cosets arising in [4] by identifying the generators introduced in [4] and by inspecting the coset parametrizations.

In section two we will introduce the algebraic outline of the symmetric space sigma model.The Cartan and the Iwasawa decompositions will be discussed.Section three is reserved for the two equivalent formulations of the symmetric space sigma model.In section four we will present the dualisation by doubling the fields and the generators,we will also discuss the local first order equations which are generalizations of the results of [3] for two different

parametrizations of the coset G/K.We will also show that the same equations may be achieved by the dualisation performed on the coset map which is different than the one used in [1,2,3].Finally in section five we will mention about the comparison of our construction with the coset realizations of [4].

2 The Scalar Coset Manifold

The scalar manifolds of all the pure and the matter coupled $N > 2$ extended supergravities in $D = 4, 5, 6, 7, 8, 9$ dimensions as well as the maximally extended supergravities in $D \leq 11$ are homogeneous spaces in the sense that they allow a transitive action of a Lie group on them.They are in the form of a coset manifold G/K where G is a real form of a non-compact semisimple Lie group and K is a maximal compact subgroup of G.

For a real semisimple Lie algebra $\mathbf{g}_0$ each maximal compactly imbedded subalgebra $\mathbf{k}_0$ corresponding to a maximal compact subgroup of the Lie group of $\mathbf{g}_0$ is an element of a Cartan decomposition [7]

$$\mathbf{g}_0 = \mathbf{k}_0 \oplus \mathbf{u}_0 \tag{2.1}$$

which is a vector space direct sum such that if $\mathbf{g}$ is the complexification of $\mathbf{g}_0$ and σ is the conjugation of $\mathbf{g}$ with respect to $\mathbf{g}_0$ then there exists a compact real form $\mathbf{g}_k$ of $\mathbf{g}$ such that

$$\sigma(\mathbf{g}_k) \subset \mathbf{g}_k \quad , \quad \mathbf{k}_0 = \mathbf{g}_0 \cap \mathbf{g}_k \quad , \quad \mathbf{u}_0 = \mathbf{g}_0 \cap (i\mathbf{g}_k). \tag{2.2}$$

The proof of the existence of Cartan decompositions can be found in [7].If we define a map $\omega : \mathbf{g}_0 \to \mathbf{g}_0$ such that $\omega(T + X) = T - X \quad (\forall T \in \mathbf{k}_0$ and $\forall X \in \mathbf{u}_0)$ then $(\mathbf{g}_0, \omega)$ is an orthogonal symmetric Lie algebra of non-compact type.The pair (G, K) associated with $(\mathbf{g}_0, \omega)$ is a Riemannian symmetric pair.Therefore K is a closed subgroup of G and the quotient topology on G/K induced by G generates a unique analytical manifold structure and G/K is a Riemannian globally symmetric space for all the possible G-invariant Riemann structures on G/K.The exponential map Exp [7] is a diffeomorphism from $\mathbf{u}_0$ onto the space G/K.Therefore there is a legitimate parametrization of the coset manifold by using $\mathbf{u}_0$.If $\{T_i\}$ are the basis vectors of $\mathbf{u}_0$ and $\{\varphi^i(x)\}$ are C^∞ maps over the D-dimensional spacetime the map

$$\nu(x) = e^{\varphi^i(x) T_i} \tag{2.3}$$

is an onto C^∞ map from the D-dimensional spacetime to the Riemannian globally symmetric space G/K.

If we define the Cartan involution θ as the involutive automorphism of $\mathbf{g}_0$ for which the bilinear form $-<X,\theta Y>$ is positive definite $\forall X,Y\in\mathbf{g}_0$ then the root space of the Cartan subalgebra (the maximal Abelian subalgebra) $\mathbf{h}_0$ of $\mathbf{g}_0$ decomposes into two orthogonal components with eigenvalues ±1 under θ [6].If we denote the set of invariant roots as Δ_c $(\theta(\alpha)=\alpha)$ then their intersection with the set of positive roots will be denoted as $\Delta_c^+=\Delta^+\cap\Delta_c$.The remaining roots in Δ^+ namely $\Delta_{nc}^+=\Delta^+-\Delta_c^+$ are the ones whose corresponding generators $\{E_\alpha\}$ do not commute with the elements of $\mathbf{h}_k=\mathbf{h}_0\cap\mathbf{u}_0$ where $\mathbf{h}_k$ is the maximal Abelian subspace of $\mathbf{u}_0$ and it consists of the non-compact part of $\mathbf{h}_0$ [5].

The Cartan subalgebra generates an Abelian subgroup in G which is called the torus [6].Although we call it torus it is not the ordinary torus topologically in fact it has the topology $(S^1)^m\times\mathbb{R}^n$ for some m and n and if it is diagonalizable in $\mathbb{R}$ (such that $m=0$) then it is called an R-split torus.These definitions can be generalized for the subalgebras of $\mathbf{h}_0$ aswell.The subspace of G which is generated by $\mathbf{h}_k$ is the maximal R-split torus in G in the sense defined above and it's dimension is called the R-rank which we will denote by r.There are two special classes of semisimple real forms.If r is maximal such that $r=l$ where l is the rank of G ($l=\dim(\mathbf{h}_0)$),which also means $\mathbf{h}_k=\mathbf{h}_0$ then the Lie group G is said to be in split real form (maximally non-compact).If on the other hand r is minimal such that $r=0$ then G is a compact real form.All the other cases in between are non-compact semisimple real forms.

Since we assume that G is a non-compact real form of a semisimple Lie group the Iwasawa decomposition [7] enables a link between the Cartan decomposition and the root space decomposition of the semisimple real lie algebra $\mathbf{g}_0$

$$\mathbf{g}_0=\mathbf{k}_0\oplus\mathbf{s}_0 \tag{2.4}$$

where $\mathbf{s}_0$ is a solvable Lie algebra of $\mathbf{g}_0$ and the sum is a direct vector space sum like in (2.1).A Lie algebra $\mathbf{g}$ is called solvable if it's n-th derived algebra is $\{0\}$ denoted as $D^n\mathbf{g}=0$ where the first derived algebra is the ideal of $\mathbf{g}$ generated by the commutator $[X,Y]$ of all of it's elements X,Y and the higher order derived algebras are defined inductively in the same way over one less rank derived algebra.Starting from the Cartan decomposition one

can show that $\mathbf{s}_0$ indeed exists and Iwasawa decomposition is possible for a non-compact real form of a semisimple Lie group [5,7].

The real Lie algebra $\mathbf{g}_0$ is a real form of the complex Lie algebra $\mathbf{g}$.We will denote the complex Lie subalgebra of $\mathbf{g}$ generated by the positive root generators $\{E_\alpha\}$ for $\alpha \in \Delta^+_{nc}$ as $\mathbf{n}$.A real Lie subalgebra of $\mathbf{g}_0$ can then be defined as $\mathbf{n}_0 = \mathbf{g}_0 \cap \mathbf{n}$.Both $\mathbf{n}$ and $\mathbf{n}_0$ are nilpotent Lie algebras.The solvable real Lie algebra $\mathbf{s}_0$ of $\mathbf{g}_0$ defined in the Iwasawa decomposition in (2.4) can then be written as a direct vector space sum [7]

$$\mathbf{s}_0 = \mathbf{h}_k \oplus \mathbf{n}_0 \tag{2.5}$$

where $\mathbf{h}_k = \mathbf{h}_0 \cap \mathbf{u}_0$ as defined before.Therefore the elements of the coset G/K are in one-to-one correspondence with the elements of $\mathbf{s}_0$ trough the exponential map as a result of (2.4).This parametrization of the coset G/K is called the solvable Lie algebra parametrization [5].

3 The Sigma Model

We will now present two equivalent formulations of the symmetric space sigma model which governs the scalar sector of a class of supergravity theories which have homogeneous coset scalar manifolds as mentioned in the previous section.The first of these formulations does not specify a coset parametrization while the second one makes use of the results of the Iwasawa decomposition.The first formulation is a more general one which is valid for any Lie group G and it's subgroup K namely it is the G/K non-linear sigma model.In particular it is applicable to the symmetric space sigma models.If we consider the set of G-valued maps $\nu(x)$ which transform onto each other as $\nu \rightarrow g\nu k(x) \quad \forall g \in G, k(x) \in K$ we can calculate $\mathcal{G} = \nu^{-1} d\nu$ which is the pull back of the Lie algebra $\mathbf{g}_0$-valued Cartan form over G through the map $\nu(x)$.The map $\nu(x)$ can always be chosen to be a parametrization of the coset G/K.Moreover if G obeys the conditions presented in the last section $\nu(x)$ can be taken as the map (2.3) $\nu(x) = e^{\varphi^i(x) T_i}$ by using the Cartan decomposition.As it is clear from section two when G is a real form of a non-compact semisimple Lie group we can function the Iwasawa decomposition resulting in the solvable Lie algebra parametrization aswell.For the most general case of $\nu(x)$ we have

$$\mathcal{G}_\mu dx^\mu = (f^a_\mu(x) T_a + \omega^i_\mu(x) K_i) dx^\mu \tag{3.1}$$

where $T_a \in \mathbf{u}_0$ and $K_i \in \mathbf{k}_0$.Here $\mathbf{u}_0$ is the orthogonal complement of $\mathbf{k}_0$ in $\mathbf{g}_0$.In particular if G is a real form of a semisimple Lie group and K is a maximal compact subgroup of G then the Cartan decomposition (2.1) can be used.When G/K is a Riemannian globally symmetric space then the fields $\{f^a_\mu\}$ form a vielbein of the G-invariant Riemann structures on G/K and $\{\omega^i_\mu\}$ can be considered as the components of the connection one forms of the gauge theory over the K-bundle.We should also bear in mind that $[\mathbf{k}_0, \mathbf{k}_0] \subset \mathbf{k}_0$ and if furthermore $[\mathbf{k}_0, \mathbf{u}_0] \subset \mathbf{u}_0$ then we will have a simpler theory.Let $P_\mu \equiv f^a_\mu T_a$ and $Q_\mu \equiv \omega^i_\mu K_i$ then we can construct a Lagrangian [8,9]

$$\mathcal{L} = \frac{1}{2} tr(P_\mu P^\mu) \tag{3.2}$$

where the trace is over the representation chosen.$\mathcal{L}$ is invariant when $\nu(x)$ is transformed under the rigid (global) action of G from the left and the local action of K from the right as given above.The elements of (3.1) P_μ and Q_μ are invariant under the rigid action of G but under the local action of K they transform as

$$\begin{aligned} Q_\mu &\rightarrow k(x) Q_\mu k^{-1}(x) + k(x) \partial_\mu k^{-1}(x) \\ P_\mu &\rightarrow k(x) P_\mu k^{-1}(x). \end{aligned} \tag{3.3}$$

The field equations corresponding to (3.2) are

$$\begin{aligned} D_\mu P^\mu &= \partial_\mu P^\mu + [Q_\mu, P^\mu] \\ &= 0 \end{aligned} \tag{3.4}$$

where we have introduced the covariant derivative $D_\mu \equiv \partial_\mu + [Q_\mu, \quad]$.

Besides having more general applications the above formalism covers the symmetric space sigma model in it.We will now introduce another parametrization of the coset G/K which is locally legitimate [7] as a result of the solvable Lie algebra parametrization of the Iwasawa decomposition which is discussed in the last section.When G is a real form of a non-compact semisimple Lie group and K is it's maximal compact subgroup the coset G/K can locally

be parametrized as

$$\nu(x) = \mathbf{g}_H(x)\mathbf{g}_N(x)$$
$$= e^{\frac{1}{2}\phi^i(x)H_i}e^{\chi^m(x)E_m} \tag{3.5}$$

where $\{H_i\}$ is a basis for $\mathbf{h}_k$ for $i = 1, ..., r$ and $\{E_m\}$ is a basis for $\mathbf{n}_0$.The map (3.5) is obtained by considering a map from a coordinate chart of the spacetime onto a neighborhood of the identity of $\mathbf{s}_0$.Thus it defines locally a diffeomorphism from $\mathbf{h}_k \times \mathbf{n}_0$ into the space G/K so it is a local parametrization of G/K.We assume that the locality is both over the spacetime and over the space $\mathbf{h}_k \times \mathbf{n}_0$ in order to write two products of exponentials instead of the solvable Lie algebra parametrization.If G is in split real form then $\mathbf{h}_k = \mathbf{h}_0$ and $\Delta^+_{nc} = \Delta^+$ thus the solvable algebra $\mathbf{s}_0$ becomes the Borel subalgebra of $\mathbf{g}_0$.The fields $\{\phi^i\}$ are called the dilatons and $\{\chi^m\}$ are called the axions.At this stage we can calculate the field equations (3.4) in terms of these newly defined fields under the parametrization (3.5).The Cartan form $\mathcal{G} = \nu^{-1}d\nu$ can be calculated from (3.5) as follows

$$\mathcal{G} = \nu^{-1}d\nu$$
$$= (\mathbf{g}_N^{-1}\mathbf{g}_H^{-1})(d\mathbf{g}_H\mathbf{g}_N + \mathbf{g}_H d\mathbf{g}_N) \tag{3.6}$$
$$= \mathbf{g}_N^{-1}d\mathbf{g}_N + \mathbf{g}_N^{-1}\mathbf{g}_H^{-1}d\mathbf{g}_H\mathbf{g}_N.$$

If we make use of the identity $e^{-C}de^{C} = dC - \frac{1}{2!}[C, dC] + \frac{1}{3!}[C, [C, dC]] - ....$

in a matrix representation the first term can be calculated as

$$
\begin{aligned}
\mathbf{g}_N^{-1} d\mathbf{g}_N &= e^{-\chi^m E_m} d e^{\chi^m E_m} \\
&= d\chi^m E_m - \frac{1}{2!}[\chi^m E_m, d\chi^n E_n] + \frac{1}{3!}[\chi^m E_m, [\chi^l E_l, d\chi^n E_n]] - .... \\
&= d\chi^m E_m - \frac{1}{2!}\chi^m d\chi^n K^v_{mn} E_v + \frac{1}{3!}\chi^m \chi^l d\chi^n K^v_{ln} K^u_{mv} E_u - .... \\
&= \overrightarrow{\mathbf{E}}\, \mathbf{\Sigma}\, \overrightarrow{d\chi}
\end{aligned}
\tag{3.7}
$$

where we have defined the row vector $(\overrightarrow{\mathbf{E}})_\alpha = E_\alpha$ and the column vector $(\overrightarrow{d\chi})^\alpha = d\chi^\alpha$.We have also introduced $\mathbf{\Sigma}$ as the $\dim\mathbf{n}_0 \times \dim\mathbf{n}_0$ matrix

$$
\mathbf{\Sigma} = \sum_{n=0}^{\infty} \frac{(-1)^n \omega^n}{(n+1)!}
\tag{3.8}
$$

$\omega^\gamma_\beta = \chi^\alpha\, K^\gamma_{\alpha\beta}$ where the structure constants $K^\gamma_{\alpha\beta}$ are defined as $[E_\alpha, E_\beta] = K^\gamma_{\alpha\beta}\, E_\gamma$.If we consider the commutator $[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}$ then $K^\alpha_{\beta\beta} = 0$ also $K^\alpha_{\beta\gamma} = N_{\beta,\gamma}$ if in the root sense $\beta + \gamma = \alpha$ and $K^\alpha_{\beta\gamma} = 0$ if $\beta + \gamma \neq \alpha$.Similarly since the Cartan generators commute with each other we have

$$
\begin{aligned}
\mathbf{g}_H^{-1} d\mathbf{g}_H &= e^{-\frac{1}{2}\phi^i H_i} d e^{\frac{1}{2}\phi^i H_i} \\
&= \frac{1}{2} d\phi^i H_i.
\end{aligned}
\tag{3.9}
$$

The second term in (3.6) can now be calculated as

$$\begin{aligned}
\mathbf{g}_N^{-1}(\frac{1}{2}d\phi^i H_i)\mathbf{g}_N &= e^{-\chi^m E_m}(\frac{1}{2}d\phi^i H_i)e^{\chi^m E_m} \\
&= \frac{1}{2}d\phi^i H_i - [\chi^m E_m, \frac{1}{2}d\phi^i H_i] \\
&+ \frac{1}{2!}[\chi^m E_m, [\chi^l E_l, \frac{1}{2}d\phi^i H_i]] - .... \\
&= \frac{1}{2}d\phi^i H_i + \chi^m \frac{1}{2}d\phi^i m_i E_m \\
&- \frac{1}{2!}\chi^m \chi^l \frac{1}{2}d\phi^i l_i K^u_{ml} E_u + ... \\
&= \frac{1}{2}d\phi^i H_i + \overrightarrow{\mathbf{E}}\,\mathbf{\Sigma}\,\overrightarrow{U}
\end{aligned} \tag{3.10}$$

where we have used the Campbell-Hausdorff formula $e^{-X}Ye^X = Y - [X,Y] + \frac{1}{2!}[X,[X,Y]] - ....$ and we have defined the column vector $(\overrightarrow{U})^m = \frac{1}{2}\chi^m m_i d\phi^i$.Also we have $[H_i, E_m] = m_i E_m$.Therefore the Cartan form $\mathcal{G} = \nu^{-1}d\nu$ in (3.6) becomes

$$\mathcal{G} = \frac{1}{2}d\phi^i H_i + \overrightarrow{\mathbf{E}}\,\mathbf{\Sigma}\,(\overrightarrow{U} + \overrightarrow{d\chi}). \tag{3.11}$$

Since the expansion of $\mathcal{G}$ consists of only the generators of $\mathbf{s}_0$ but not the generators of $\mathbf{k}_0$ which is a direct result of (3.5) where the parametrization is derived locally from the solvable Lie algebra parametrization we have $Q_\mu = 0$ and from (3.11) P_μ is

$$P_\mu = \frac{1}{2}\partial_\mu \phi^i H_i + \Sigma^\alpha_m(\frac{1}{2}\chi^m m_i \partial_\mu \phi^i + \partial_\mu \chi^m)E_\alpha. \tag{3.12}$$

Since $Q_\mu = 0$ from (3.4) the equations of motion become

$$\partial^\mu P_\mu = 0. \tag{3.13}$$

Thus we have

$$\partial^{\mu}\partial_{\mu}\phi^{i}=0$$

$$\partial^{\mu}\left(\Sigma_{m}^{\alpha}\left(\frac{1}{2}\chi^{m}m_{i}\partial_{\mu}\phi^{i}+\partial_{\mu}\chi^{m}\right)\right)=0 \tag{3.14}$$

for $i=1,...r$ and $\alpha\in\Delta_{nc}^{+}$.

Another formulation of the G/K symmetric space sigma model (with G not necessarily split) can be done by introducing the internal metric $\mathcal{M}$.In this formulation the Lagrangian which is invariant under the global action of G from the left and the local action of K from the right is [1,2,10]

$$\mathcal{L}=\frac{1}{4}tr(*d\mathcal{M}^{-1}\wedge d\mathcal{M}) \tag{3.15}$$

where the internal metric $\mathcal{M}$ is defined as $\mathcal{M}=\nu^{\#}\nu$ and $\#$ is the generalized transpose over the Lie group G such that $(exp(g))^{\#}=exp(g^{\#})$.It is induced by the Cartan involution θ over $\mathbf{g}_0$ ($g^{\#}=-\theta(g)$) [1,10].The Lagrangian can be expressed as

$$\mathcal{L}=-\frac{1}{2}tr(*d\nu\nu^{-1}\wedge(d\nu\nu^{-1})^{\#}+*d\nu\nu^{-1}\wedge d\nu\nu^{-1}). \tag{3.16}$$

We will again assume the parametrization given in (3.5).By following the same steps given in detail in [3] we can calculate the $\mathbf{s}_0$-valued one form $\mathcal{G}_0=d\nu\nu^{-1}$.This is possible because the Borel parametrization used in [3] is a limit case of the solvable Lie algebra parametrization as discussed in section two so as far as the commutation relations are concerned the equivalence between the general case and the split case of [3] is straightforward.Thus from (3.5) we have

$$\begin{aligned}\mathcal{G}_0&=d\nu\nu^{-1}\\&=\frac{1}{2}d\phi^{i}H_{i}+e^{\frac{1}{2}\alpha_{i}\phi^{i}}F^{\alpha}E_{\alpha}\\&=\frac{1}{2}d\phi^{i}H_{i}+\overset{\rightharpoonup}{\mathbf{E}'}\,\mathbf{\Omega}\,\overset{\rightharpoonup}{d\chi}.\end{aligned} \tag{3.17}$$

where $\{H_i\}$ for $i = 1, ..., r$ are the generators of $\mathbf{h}_k$ and $\{E_\alpha\}$ for $\alpha \in \Delta^+_{nc}$ are the generators of $\mathbf{n}_0$.We have defined $F^\alpha = \mathbf{\Omega}^\alpha_\beta d\chi^\beta$ and the row vector $(\overrightarrow{\mathbf{E}'})_\alpha = e^{\frac{1}{2}\alpha_i\phi^i}E_\alpha$.Also $\mathbf{\Omega}$ is a $\dim\mathbf{n}_0\times\dim\mathbf{n}_0$ matrix

$$\begin{aligned}\mathbf{\Omega} &= \sum_{n=0}^{\infty}\frac{\omega^n}{(n+1)!}\\ &= (e^\omega - I)\,\omega^{-1}\end{aligned} \tag{3.18}$$

The matrix ω has been already defined before.The equations of motion of the Lagrangian (3.15) can be found as [6,10]

$$\begin{aligned} d(*d\phi^i) &= \frac{1}{2}\sum_{\alpha\in\Delta^+_{nc}}\alpha_i e^{\frac{1}{2}\alpha_i\phi^i}F^\alpha \wedge e^{\frac{1}{2}\alpha_i\phi^i} * F^\alpha \\ d(e^{\frac{1}{2}\gamma_i\phi^i} * F^\gamma) &= -\frac{1}{2}\gamma_j e^{\frac{1}{2}\gamma_i\phi^i}d\phi^j \wedge *F^\gamma \\ &\quad + \sum_{\alpha-\beta=-\gamma} e^{\frac{1}{2}\alpha_i\phi^i}e^{\frac{1}{2}\beta_i\phi^i}N_{\alpha,-\beta}F^\alpha \wedge *F^\beta \end{aligned} \tag{3.19}$$

where $i, j = 1, ..., r$ and $\alpha, \beta, \gamma \in \Delta^+_{nc}$.We have put the second equation in a convenient form for the analysis we will use in the dualisation section as in [3].

We will now introduce a transformation between the two parametrizations given in (2.3) and (3.5) which are based on two different sets of scalar functions.We can derive a procedure to calculate the transformation between these two sets.We may assume that the coset valued maps in (2.3) and (3.5) can be chosen to be equal.This is possible if we restrict the scalar maps with the ones which generate ranges in sufficiently small neighborhoods around the identity element of $\mathbf{g}_0$ when they are coupled to the algebra generators in (2.3) and (3.5) [7].This local equality is sufficient since our aim is to obtain the local first order formulation of the parametrization of (2.3) from the first order formulation which is based on (3.5) in the next section.We will firstly show a method through which one can calculate the exact transformations from $\{\varphi^i\}$ to $\{\phi^j, \chi^m\}$.We will not attempt to solve the explicit transformation functions which are dependent on the structure constants in

a complicated way.One can solve these set of differential equations when the structure constants are specified.Let us first define the function

$$f(\lambda) = e^{\lambda(\frac{1}{2}\phi^i H_i)} e^{\lambda(\chi^\alpha E_\alpha)}. \tag{3.20}$$

Taking the derivative of $f(\lambda)$ with respect to λ gives

$$\frac{\partial f(\lambda)}{\partial \lambda} f^{-1}(\lambda) = \frac{1}{2}\phi^i H_i + e^{\frac{\lambda}{2}\phi^i \alpha_i}\chi^\alpha E_\alpha. \tag{3.21}$$

We have used the fact that $e^{\frac{\lambda}{2}\phi^i H_i}\chi^\alpha E_\alpha e^{-\frac{\lambda}{2}\phi^i H_i} = e^{\frac{\lambda}{2}\phi^i \alpha_i}\chi^\alpha E_\alpha$.Now if we let $f(\lambda) = e^{C(\lambda)}$ where we define $C(\lambda) = \varphi^i(\lambda) T_i$ and use the formula $de^C e^{-C} = dC + \frac{1}{2!}[C, dC] + \frac{1}{3!}[C,[C,dC]] + ...$ we find that

$$\frac{1}{2}\phi^i H_i + e^{\frac{\lambda}{2}\phi^i \alpha_i}\chi^\alpha E_\alpha = \vec{\mathbf{T}}\,\mathbf{S}(\lambda)\,\vec{\partial\varphi} \tag{3.22}$$

where the components of the row vector $\vec{\mathbf{T}}$ are $T_i = H_i$ for $i = 1, ..., r$ and $T_{\alpha+r} = E_\alpha$ for $\alpha = 1, ...,\dim\mathbf{n}_0$.Besides the column vector $\vec{\partial\varphi}$ is defined as $\{\frac{\partial\varphi^i(\lambda)}{\partial\lambda}\}$.We have also introduced the $\dim\mathbf{s}_0 \times \dim\mathbf{s}_0$ matrix $\mathbf{S}(\lambda)$ as

$$\begin{aligned} \mathbf{S}(\lambda) &= \sum_{n=0}^{\infty} \frac{V^n(\lambda)}{(n+1)!} \\ &= (e^{V(\lambda)} - I)V^{-1}(\lambda) \end{aligned} \tag{3.23}$$

The matrix $V(\lambda)$ is $V_\alpha^\beta(\lambda) = \varphi^i(\lambda) C_{i\alpha}^\beta$ for $[T_i, T_j] = C_{ij}^k T_k$.The calculation of the right hand side of (3.22) is similar to (3.17).If the structure constants are given for a particular $\mathbf{s}_0$ one can obtain the functions $\{\varphi^i(\lambda)\}$ from the set of differential equations (3.22).Then setting $\lambda = 1$ will yield the desired set of functions $\{\varphi^i(\phi^j, \chi^\alpha)\}$.We might also make use of a direct calculation namely the Lie's theorem [7].For a matrix representation and in a neighborhood of the identity if we let $e^C = e^A e^B$ then

$$\begin{aligned} C &= B + \int_0^1 g(e^{tadA} e^{adB}) A dt \\ &= A + B + \frac{1}{2}[A, B] + \frac{1}{12}([A,[A,B]] - [B,[B,A]]) + ... \end{aligned} \tag{3.24}$$

where $g \equiv lnz/z-1$.In the above equation if we choose $A = \frac{1}{2}\phi^i H_i$,$B = \chi^\alpha E_\alpha$ and $C = \varphi^i T_i$ we can calculate the transformations we need.

We may also derive the differential form of the transformation between the two parametrizations which is more essential for our purposes of obtaining the local first order formulation of the parametrization in (2.3) in the next section.From (2.3) by choosing $\{T_i\}$ as the basis of $\mathbf{s}_0$,similar to the previous calculations we can calculate the $\mathbf{s}_0$-valued Cartan form $d\nu\nu^{-1}$ as

$$\begin{aligned} d\nu\nu^{-1} &= de^{\varphi^i T_i} e^{-\varphi^i T_i} \\ &= \overset{\rightharpoonup}{\mathbf{T}}\, \mathbf{\Delta}\, \overset{\rightharpoonup}{\mathbf{d}}\varphi. \end{aligned} \tag{3.25}$$

We have defined $\overset{\rightharpoonup}{\mathbf{T}}$ before $\overset{\rightharpoonup}{\mathbf{d}}\varphi$ is a column vector of the field strengths $\{d\varphi^i\}$ and the $\dim\mathbf{s}_0 \times \dim\mathbf{s}_0$ matrix $\mathbf{\Delta}$ can be given as

$$\begin{aligned} \mathbf{\Delta} &= \sum_{n=0}^{\infty} \frac{M^n}{(n+1)!} \\ &= (e^M - I)M^{-1} \end{aligned} \tag{3.26}$$

where $M_\alpha^\beta = \varphi^i C_{i\alpha}^\beta$.We should imply that $\mathbf{\Delta} = \mathbf{S}(\lambda = 1)$ and $M = V(\lambda = 1)$.If we refer to the equation (3.17) we have already calculated the $\mathbf{s}_0$-valued Cartan form for the parametrization of (3.5).Therefore if we compare (3.17) and (3.25) since locally they must be equal we find

$$\begin{aligned} \mathbf{\Delta}_i^\gamma d\varphi^i &= \frac{1}{2} d\phi^\gamma \\ \mathbf{\Delta}_i^\beta d\varphi^i &= e^{\frac{1}{2}\beta_j \phi^j} \mathbf{\Omega}_k^\beta d\chi^k. \end{aligned} \tag{3.27}$$

The indices above are $\gamma = 1, ..., r$; $\beta, k = r+1, ..., \dim\mathbf{n}_0$; $i = 1, ..., \dim\mathbf{s}_0 = r+\dim\mathbf{n}_0$ and $\beta \in \Delta_{nc}^+$.As a result we have obtained the differential form of the transformation between $\{\varphi^i\}$ and $\{\phi^j, \chi^\alpha\}$.It can be seen that the relation between the two scalar parametrizations is dependent on the structure constants in a very complicated way.One can also integrate (3.27) to obtain the explicit form of this transformation as an alternative to the equation (3.22).

Finally we will calculate the field equations (3.4) for the parametrization (2.3) by assuming the Iwasawa decomposition.Smilar to (3.25) from (2.3) the $\mathbf{s}_0$-valued Cartan form $\nu^{-1}d\nu$ is

$$\begin{aligned}\nu^{-1}d\nu &= e^{-\varphi^i T_i} de^{\varphi^i T_i} \\ &= \overset{\rightharpoonup}{\mathbf{T}}\,\mathbf{W}\,\overset{\rightharpoonup}{\mathbf{d}\varphi}.\end{aligned} \tag{3.28}$$

where we have

$$\begin{aligned}\mathbf{W} &= \sum_{n=0}^{\infty} \frac{(-1)^n M^n}{(n+1)!} \\ &= (I - e^{-M})M^{-1}.\end{aligned} \tag{3.29}$$

From (3.28) like in (3.11) we see that $Q_\mu = 0$ due to the solvable Lie algebra parametrization.On the other hand P_μ is

$$\begin{aligned}P_\mu &= P^i_\mu T_i \\ &= (\mathbf{W})^{\,l}_{\,k}\, \partial_\mu \varphi^k\, T_l.\end{aligned} \tag{3.30}$$

Thus in terms of the fields $\{\varphi^i\}$ the second order equations (3.4) become

$$\begin{aligned}\partial^\mu P_\mu &= \partial^\mu \left((\mathbf{W})^{\,l}_{\,k}\, \partial_\mu \varphi^k T_l \right) \\ &= 0.\end{aligned} \tag{3.31}$$

4 Dualisation and the First Order Formulation

The local first order formulation of the G/K symmetric space sigma model when G is in split real form has been given in [3] where we have applied the standard dualisation method of [1,2] by introducing dual generators for the Borel subgroup generators and also new auxiliary fields $(D-2)$-forms for the scalar fields.Then the enlarged Lie superalgebra which contains the original

Borel algebra has been inspected so that it would realize the original second order equations in an enlarged coset model.In [3] after calculating the extra commutation relations coming from the new generators locally the first order equations were given as a twisted self-duality equation $*\mathcal{G}' = \mathcal{S}\mathcal{G}'$ where $\mathcal{G}'$ is the doubled field strength (the Cartan form generated by the new coset representative) and $\mathcal{S}$ is a pseudo-involution of the enlarged Lie superalgebra which for the special case of the scalar coset maps the original generators onto the dual ones and the dual generators onto the original scalar generators with a sign factor depending on the dimension D and the signature s of the spacetime as explained in [1,2,3].The split rigid group symmetric space sigma model is a limiting case as discussed in section two.The solvable algebra is a subalgebra of the Borel algebra in general and the field equations (3.19) for the general non-compact (not necessarily split) real form model are in the same form with the split case except the summing indices.Therefore the results in [3] can be generalized for the general non-compact symmetric space sigma model.We will give a summary of the results which are generalizations and whose detailed calculations are similar to the ones in [3].

Firstly we will introduce dual $(D-2)$-form fields for the dilatons and the axions which are defined in (3.5).The dual fields will be denoted as $\{\widetilde{\phi^i}\}$ and $\{\widetilde{\chi^m}\}$.For each scalar generator we will also define dual generators which will extend the solvable Lie algebra $\mathbf{s}_0$ to a Lie superalgebra which generates a differential algebra with the local differential form algebra [2].These generators are $\{\widetilde{E}_m\}$ as duals of $\{E_m\}$ and $\{\widetilde{H}_i\}$ for $\{H_i\}$.If we define a new parametrization into the enlarged group

$$\nu'(x) = e^{\frac{1}{2}\phi^i H_i} e^{\chi^m E_m} e^{\widetilde{\chi}^m \widetilde{E}_m} e^{\frac{1}{2}\widetilde{\phi}^i \widetilde{H}_i} \tag{4.1}$$

we can calculate the doubled field strength $\mathcal{G}' = d\nu'\nu'^{-1}$ and then use the twisted self-duality condition $*\mathcal{G}' = \mathcal{S}\mathcal{G}'$ to find the structure constants of the dual generators and the first order equations of motion.The general form of the commutation relations in addition to the ones of $\mathbf{s}_0$ can be given as [1,2]

$$[E_\alpha, \widetilde{T}_m] = \widetilde{f}^n_{\alpha m} \widetilde{T}_n \quad , \quad [H_i, \widetilde{T}_m] = \widetilde{g}^n_{im} \widetilde{T}_n,$$

$$[\widetilde{T}_m, \widetilde{T}_n] = 0 \tag{4.2}$$

where $\widetilde{T}_i = \widetilde{H}_i$ for $i = 1, ..., r$ and $\widetilde{T}_{\alpha+r} = \widetilde{E}_\alpha$ for $\alpha = 1, ...,\dim\mathbf{n}_0$.In general the pseudo-involution $\mathcal{S}$ maps the original generators onto the dual

ones and it has the same eigenvalues ± 1 with the action of the operator $*\circ *$ on the corresponding dual field strength of the coupling dual potential.Therefore $\mathcal{S}T_i = \widetilde{T}_i$ and $\mathcal{S}\widetilde{T}_i = (-1)^{(p(D-p)+s)}T_i$ where p is the degree of the dual field strength and s is the signature of the spacetime.The degree of the dual field strengths corresponding to the dual generators is $(D-1)$ in our case.In [3] the signature of the spacetime is assumed to be $(D-1)$ for this reason $\mathcal{S}$ sends the dual generators to the scalar ones with a positive sign.In this work we will assume that the signature is $s = 1$.Thus the sign factor above is dependent on the spacetime dimension D and we have $\mathcal{S}\widetilde{T}_i = (-1)^D T_i$.Now by following the same steps in [1,2,3] and using the twisted self-duality equation we can express the doubled field strength as

$$\mathcal{G}' = d\nu\nu^{-1} + \frac{1}{2}(-1)^D * d\phi^i \widetilde{H}_i + (-1)^D e^{\frac{1}{2}\alpha_i\phi^i} * F^\alpha \widetilde{E}_\alpha. \tag{4.3}$$

The Cartan form $\mathcal{G}_0 = d\nu\nu^{-1}$ is already calculated in (3.17).As explained in [3] the generators $\{T_i\}$ are even and $\{\widetilde{T}_i\}$ are even or odd depending on the spacetime dimension D within the context of the differential algebra generated by the solvable algebra generators, their duals and the differential forms.By using the properties of this differential algebra [2] and the fact that from it's definition $\mathcal{G}'$ obeys the Cartan-Maurer equation

$$d\mathcal{G}' - \mathcal{G}' \wedge \mathcal{G}' = 0 \tag{4.4}$$

we can show that if we choose

$$[T_i, \widetilde{H}_j] = 0,$$

$$[H_j, \widetilde{E}_\alpha] = -\alpha_j \widetilde{E}_\alpha \quad , \quad [E_\alpha, \widetilde{E}_\alpha] = \frac{1}{4}\sum_{j=1}^{r} \alpha_j \widetilde{H}_j,$$

$$[E_\alpha, \widetilde{E}_\beta] = N_{\alpha,-\beta}\widetilde{E}_\gamma, \qquad \alpha - \beta = -\gamma, \alpha \neq \beta. \tag{4.5}$$

for $i = 1, ..., \dim \mathbf{s}_0, j = 1, ..., r$ and $\alpha, \beta.\gamma \in \Delta^+_{nc}$ then (4.4) by inserting (4.3) gives the correct second order equations (3.19).As a matter of fact if we choose in general $[E_\alpha, \widetilde{H}_j] = a_{\alpha j}\alpha_j \widetilde{E}_\alpha$ and $[H_j, \widetilde{E}_\alpha] = b_{j\alpha}\alpha_j \widetilde{E}_\alpha$ with $a_{j\alpha}, b_{\alpha j}$ arbitrary but obeying the constraint $a_{j\alpha} + b_{\alpha j} = -1$ in addition to the

rest of the commutators in (4.5) we can satisfy the second order equations (3.19).However for simplicity like in [3] we will choose $a_{j\alpha}=0$ as seen in (4.5).Therefore the structure constants in (4.2) become

$$\widetilde{f}^{n}_{\alpha m}=0, \qquad m\leq r \quad , \quad \widetilde{f}^{i}_{\alpha,\alpha+r}=\frac{1}{4}\alpha_i, \qquad i\leq r$$

$$\widetilde{f}^{i}_{\alpha,\alpha+r}=0, \qquad i>r \quad , \quad \widetilde{f}^{i}_{\alpha,\beta+r}=0, \qquad i\leq r,\alpha\neq\beta$$

$$\widetilde{f}^{\gamma+r}_{\alpha,\beta+r}=N_{\alpha,-\beta}, \qquad \alpha-\beta=-\gamma,\alpha\neq\beta$$

$$\widetilde{f}^{\gamma+r}_{\alpha,\beta+r}=0, \qquad \alpha-\beta\neq-\gamma,\alpha\neq\beta. \tag{4.6}$$

Also

$$\widetilde{g}^{n}_{im}=0, \qquad m\leq r \quad , \quad \widetilde{g}^{n}_{im}=0, \qquad m>r,m\neq n$$

$$\widetilde{g}^{\alpha}_{i\alpha}=-\alpha_i, \qquad \alpha>r. \tag{4.7}$$

These relations with the commutation relations of the solvable Lie algebra $\mathbf{s}_0$ form the complete algebraic structure of the enlarged Lie superalgebra.In (4.3) $\mathcal{G}'$ has been given only in terms of the original scalar fields as the twisted self-duality condition has been used primarily.From the definition of $\nu'(x)$ in (4.1) since we have obtained the full set of commutation relations without using the twisted self-duality condition we can explicitly calculate the Cartan form $\mathcal{G}'$ in terms of both the scalar fields and their duals [3].

$$\begin{aligned}\mathcal{G}' &= d\nu'\nu'^{-1} \\ &= \frac{1}{2}d\phi^i H_i+\overrightarrow{\mathbf{E}'}\,\mathbf{\Omega}\,\overrightarrow{d\chi}+\overrightarrow{\widetilde{\mathbf{T}}}\,e^{\mathbf{\Gamma}}\,e^{\mathbf{\Lambda}}\,\overrightarrow{\mathbf{A}}.\end{aligned} \tag{4.8}$$

We have introduced the matrices $(\Gamma)^k_n=\frac{1}{2}\phi^i\,\widetilde{g}^k_{in}$ and $(\Lambda)^k_n=\chi^m\,\widetilde{f}^k_{mn}$.The row vector $\overrightarrow{\widetilde{\mathbf{T}}}$ is defined as $\{\widetilde{\mathbf{T}}^i\}$ and the column vector $\overrightarrow{\mathbf{A}}$ is $\mathbf{A}^i=\frac{1}{2}d\widetilde{\phi}^i$ for $i=1,...,r$ and $\mathbf{A}^{\alpha+r}=d\widetilde{\chi}^\alpha$,$\alpha\in\Delta^+_{nc}$ in other words if we enumerate the roots

in Δ^{+}_{nc} then $\alpha = 1, ..., \dim\mathbf{n}_0$.When we apply the twisted self-duality condition $*\mathcal{G}' = \mathcal{S}\mathcal{G}'$ on (4.8) we may achieve the first order equations locally whose exterior derivative will give the second order equations (3.19), [1,2].Therefore similar to the split case in [3] the locally integrated first order field equations of the Lagrangian (3.15) are

$$*\overrightarrow{\mathbf{\Psi}} = (-1)^{D} e^{\mathbf{\Gamma}} e^{\mathbf{\Lambda}} \overrightarrow{\mathbf{A}}. \tag{4.9}$$

The column vector $\overrightarrow{\mathbf{\Psi}}$ is defined as $\mathbf{\Psi}^{i} = \frac{1}{2} d\phi^{i}$ for $i = 1, ...r$ and $\mathbf{\Psi}^{\alpha+r} = e^{\frac{1}{2}\alpha_i\phi^i} \mathbf{\Omega}^{\alpha}_{l} d\chi^{l}$ where $\alpha = 1, ..., \dim\mathbf{n}_0$.Due to the assumed signature of the spacetime these equations unlike the ones in [3] have a sign factor depending on the spacetime dimension.We should also observe that the case $SL(2, \mathbb{R})$ $/SO(2)$ scalar coset of IIB supergravity whose first-order equations are explicitly calculated in [3] has a positive sign factor for both of the signatures introduced here and in [3] since $D = 10$.

Following the discussion in section three we can also find the first order equations for the set $\{\varphi^{i}\}$.Firstly we can define the transformed matrices $\mathbf{\Gamma}'(\varphi^{j}) \equiv \mathbf{\Gamma}(\phi^{i}(\varphi^{j}))$ and $\mathbf{\Lambda}'(\varphi^{j}) \equiv \mathbf{\Lambda}(\chi^{m}(\varphi^{j}))$ which can be obtained by calculating the local transformation rules from (3.22) or (3.27).If we make the observation that the right hand side of the differential form of the transformation between $\{\varphi^{i}\}$ and $\{\phi^{i}, \chi^{\alpha}\}$ namely (3.27) are the components of $\overrightarrow{\mathbf{\Psi}}$,from (4.9) we can write down the first order equations for $\{\varphi^{i}\}$ as

$$\mathbf{\Delta} * \overrightarrow{\mathbf{d}\varphi} = (-1)^{D} e^{\mathbf{\Gamma}'} e^{\mathbf{\Lambda}'} \overrightarrow{\mathbf{A}}. \tag{4.10}$$

We may also transform (4.10) so that we do not have to calculate the explicit transformations between the fields.Firstly we should observe that the structure constants $\{\widetilde{g}^{k}_{in}\}$ and $\{\widetilde{f}^{k}_{mn}\}$ of (4.2) form a representation for $\mathbf{s}_0$ as the representatives of the basis $\{H_i\}$ and $\{E_m\}$ respectively.Thus under the representation

$$\begin{aligned} e^{\frac{1}{2}\phi^{i}H_{i}} e^{\chi^{m}E_{m}} &= e^{\mathbf{\Gamma}} e^{\mathbf{\Lambda}} \\ &\equiv e^{\frac{1}{2}\phi^{i}\widetilde{g}^{k}_{in}} e^{\chi^{m}\widetilde{f}^{k}_{mn}}. \end{aligned} \tag{4.11}$$

In section three we have assumed that locally

$$e^{\varphi^{i}T_{i}} = e^{\frac{1}{2}\phi^{i}H_{i}} e^{\chi^{m}E_{m}}. \tag{4.12}$$

Therefore the first-order equations (4.10) for $\{\varphi_i\}$ can be written as

$$\mathbf{\Delta} * \overrightarrow{\mathbf{d}\varphi} = (-1)^D e^{\mathbf{\Pi}} \overrightarrow{\mathbf{A}} \tag{4.13}$$

where we have defined $(\mathbf{\Pi})^k_n = \sum\limits_{i=1}^{r} \varphi^i \widetilde{g}^k_{in} + \sum\limits_{m=r+1}^{dim\mathbf{n}_0} \varphi^m \widetilde{f}^k_{mn}$.by using the representation established by (4.2).

One may also obtain these first order equations independently by applying the dualisation method on the parametrization (2.3).We again assume the solvable Lie algebra parametrization.Let us first define the doubled coset map

$$\nu'' = e^{\varphi^i T_i} e^{\widetilde{\varphi}^i \widetilde{T}_i} \tag{4.14}$$

in which we have introduced the dual fields and generators as usual.If we calculate the Cartan form $\mathcal{G}'' = d\nu'' \nu''^{-1}$ by carrying out similar calculations like we have done before we find that

$$\begin{aligned} \mathcal{G}'' &= de^{\varphi^i T_i} e^{-\varphi^i T_i} + e^{\varphi^i T_i} de^{\widetilde{\varphi}^i \widetilde{T}_i} e^{-\widetilde{\varphi}^i \widetilde{T}_i} e^{-\varphi^i T_i} \\ &= \overrightarrow{\mathbf{T}}\, \mathbf{\Delta}\, \overrightarrow{\mathbf{d}\varphi} + \overrightarrow{\widetilde{\mathbf{T}}}\, e^{\mathbf{\Pi}}\, \overrightarrow{\mathbf{d}\widetilde{\varphi}}. \end{aligned} \tag{4.15}$$

The first term has already been calculated in (3.25).We have calculated the structure constants related to the dual generators in (4.6) and (4.7).If we apply the twisted self-duality equation $*\mathcal{G}'' = \mathcal{S}\mathcal{G}''$ above we find the first order equations as

$$\mathbf{\Delta} * \overrightarrow{\mathbf{d}\varphi} = (-1)^D e^{\mathbf{\Pi}} \overrightarrow{\mathbf{d}\widetilde{\varphi}}. \tag{4.16}$$

Since the dual fields are auxiliary fields we can always choose $(\mathbf{d}\widetilde{\varphi})^i = \mathbf{A}^i$ thus the equations (4.13) and (4.16) are the same equations.This result verifies the validity of (4.13) which is obtained by using the transformation law (3.27) in (4.9).

Finally we should point out the fact that the case when the global symmetry group G is in split real form which is analyzed in detail in [3] can be obtained by choosing $r = l$ (the rank of G) and $\Delta^+_{nc} = \Delta^+$ in the expressions given in this section.

5 The $O(p,q)/(O(p)\times O(q))$ Scalar Cosets of the $T^{10-D}-$Compactified Heterotic String Theory

In this section we will briefly discuss the correspondence between our results and the scalar cosets constructed in [4] by identifying the solvable Lie algebra generators and by comparing the coset parametrizations.The Kaluza-Klein compactification of the Bosonic sector of the ten dimensional simple supergravity which is coupled to N Abelian gauge multiplets on the Euclidean Tori T^{10-D} is discussed in [4].The scalar sectors of the resulting D-dimensional theories are formulated by the introduction of the G/K coset spaces.When as a special case the number of U(1) gauge fields is chosen to be 16 the formulation corresponds to the dimensional reduction of the low-energy effective Bosonic Lagrangian of the ten dimensional heterotic string theory.The global symmetries of the Bosonic sectors of these reduced theories are also studied in detail in [4].

As it is clear from the mainline of [4] the scalar Lagrangian of the D-dimensional compactified theories for $9 \geq D \geq 3$ can be described in the form of (3.15) with an additional dilatonic kinetic term after certain field redefinitions.It is also shown that the G/K coset representative ν and the internal metric $\mathcal{M} = \nu^T \nu$ are elements of $O(10-D, 10-D+N)$.The determination of the fiducial point W_0 =diag$(1,1,...,1)$ by choosing all the scalar fields in the coset representative ν zero enables the identification of the isotropy group as $O(10-D)\times O(10-D+N)$.Therefore the scalar manifold for the D-dimensional compactified theory with N gauge multiplet couplings becomes

$$\frac{O(10-D, 10-D+N)}{O(10-D)\times O(10-D+N)} \times \mathbb{R}. \tag{5.1}$$

The extra $\mathbb{R}$ factor arises since there is an additional dilaton which is decoupled from the rest of the scalars in the scalar Lagrangian.In [4] it is also shown that $O(10-D, 10-D+N)\times\mathbb{R}$ is the global symmetry of not only the scalar Lagrangian but the entire D-dimensional Bosonic Lagrangian aswell.Here again $\mathbb{R}$ corresponds to the constant shift symmetry of the decoupled dilaton.Furthermore the $D=4$ and the $D=3$ cases are studied separately in [4] since they have symmetry enhancements in addition to the

general scheme of (5.1).When the two-form potential is dualised with an additional axion in $D = 4$,an axion-dilaton $SL(2,\mathbb{R})$ system [1,2,3] which is decoupled from the rest of the scalars occurs in the scalar Lagrangian and the enlarged $D = 4$ scalar manifold becomes

$$\frac{O(6,6+N)}{O(6)\times O(6+N)}\times\frac{SL(2,\mathbb{R})}{O(2)}. \tag{5.2}$$

On the other hand in $D = 3$ apart from the original scalars the remaining Bosonic fields are dualised to give $7 + 7 + N$ additional axions so that the entire Bosonic Lagrangian is composed of only the scalars.The $D = 3$ scalar manifold then becomes

$$\frac{O(8,8+N)}{O(8)\times O(8+N)}. \tag{5.3}$$

We see that all of the global symmetry groups in (5.1),(5.2),(5.3) apart from the contributions of the decoupled scalars namely $O(10-D,10-D+N), O(6,6+N), O(8,8+N)$ are real forms of a non-compact semisimple Lie group and they enable solvable Lie algebra parametrizations of the cosets generated by the denominator groups $O(10-D)\times O(10-D+N), O(6)\times O(6+N), O(8)\times O(8+N)$ respectively.In fact the orthogonal algebras o(p,q) are elements of the D_n-series when $p+q = 2n$ and the B_n-series when $p+q = 2n+1$.Depending on the values of p and q the group $O(p,q)$ can be in split real form or not.Forexample $O(2,3)$ is in split real form.In the direction of the observation mentioned above the analysis of the previous sections can be applied for the scalar cosets of (5.1),(5.2),(5.3).However we should inspect the realizations of these cosets in [4] from a closer point of view.It is shown in [4] that if one assigns the set of generators $\{H^i, E_i^j, V^{ij}, U_I^i\}$ to the scalar fields $\{\phi_i, A^i_{(0)j}, A_{(0)ij}, B^I_{(0)i}\}$ resulting from the dimensional reduction in [4] respectively,the scalar Lagrangian of the compactified D-dimensional theory apart from the decoupled scalars,for all of the cases described above can be constructed in the form of (3.15) by using the coset parametrization

$$\nu = e^{\frac{1}{2}\phi_i H^i} e^{A^i_{(0)j}E_i^j} e^{\frac{1}{2}A_{(0)ij}V^{ij}} e^{B^I_{(0)i}U^i_I}. \tag{5.4}$$

The non-vanishing commutators of the generators are calculated in order that they lead to the scalar Lagrangian in [4] as

$$[\vec{H}, E_i^j] = \vec{\mathbf{b}}_{ij} E_i^j \quad , \quad [\vec{H}, V^{ij}] = \vec{\mathbf{a}}_{ij} V^{ij} \quad , \quad [\vec{H}, U_I^i] = \vec{\mathbf{c}}_i U_I^i,$$

$$[E_i^j, E_k^l] = \delta_k^j E_i^l - \delta_i^l E_k^j \quad , \quad [E_i^j, V^{kl}] = -\delta_i^k V^{jl} - \delta_i^l V^{kj},$$

$$[E_i^j, U_I^k] = -\delta_i^k U_I^j \quad , \quad [U_I^i, U_J^j] = \delta_{IJ} V^{ij} \tag{5.5}$$

where $\vec{\mathbf{a}}_{ij}, \vec{\mathbf{b}}_{ij}, \vec{\mathbf{c}}_i$ are the dilaton vectors whose detailed descriptions can be found in [4].We note that since in the D-dimensional T^{10-D}-compactified theory the scalars are coupled to the one-form potentials which form the $(20-2D+N)$-dimensional fundamental representation of $O(10-D, 10-D+N)$,both the coset representative (5.4),the internal metric $\mathcal{M}$ and the generators in (5.5) are represented by $(20-2D+N)$-dimensional matrices.

In [4] an embedding of the algebra (5.5) into $o(10-D, 10-D+N)$ is also given as

$$H_i = (2)^{1/2} h_{\widetilde{e}_i} \quad , \quad E_i^j = E_{-\widetilde{e}_i+\widetilde{e}_j} \quad , \quad V^{ij} = E_{\widetilde{e}_i+\widetilde{e}_j},$$

$$U_{2k-1}^i = (2)^{-1/2}(E_{\widetilde{e}_i+e_{k+m}} + E_{\widetilde{e}_i-e_{k+m}}),$$

$$U_{2k}^i = i(2)^{-1/2}(E_{\widetilde{e}_i+e_{k+m}} - E_{\widetilde{e}_i-e_{k+m}}) \tag{5.6}$$

where $1 \leq i < j \leq 10-D$.We have defined $m = 10-D$ and $1 \leq k \leq [N/2]$.When N is odd in addition to (5.6) we also have

$$U_N^i = E_{\widetilde{e}_i}. \tag{5.7}$$

In (5.6) and (5.7) $\{e_i\}$ is an orthonormal basis with a representation of null entries except the i'th position which is equal to one.It is used to characterize the positive roots of $o(10-D, 10-D+N)$ [4] and $\{\widetilde{e}_i\}$ are defined as

$$\widetilde{e}_i = e_{11-D-i}, \qquad 1 \leq i \leq 10-D. \tag{5.8}$$

As it is mentioned in [4] owing to their definitions the generators $\{h_{\widetilde{e}_i}\}$ appearing in (5.6) form a basis for $\mathbf{h}_k$ and the ones $\{E_{\widetilde{e}_i\pm e_j}, E_{\widetilde{e}_i\pm\widetilde{e}_j}, E_{\widetilde{e}_i}\}$ appearing in (5.6) and (5.7) are the generators of $\mathbf{n}_0$ (the last set $\{E_{\widetilde{e}_i}\}$ is included when N is odd and excluded when N is even).Therefore we conclude by following the discussion of section two that the algebra structure presented in (5.5) corresponds to the solvable Lie algebra of $o(10-D, 10-D+N)$.We can identify the basis $\{h_{\widetilde{e}_i}, E_{\widetilde{e}_i\pm e_j}, E_{\widetilde{e}_i\pm\widetilde{e}_j}, E_{\widetilde{e}_i}\}$ with the one $\{H_i, E_\alpha\}$ which we have used in our previous analysis in sections three and four bearing in mind that the even N and the odd N cases are differing.

We observe that (5.6) and (5.7) give us the transformations between the abstract solvable Lie algebra generators we have used in the previous sections and the original generators which arise in the coset formulations of the D-dimensional compactified theories of [4].Therefore the embedding of the generators of (5.4) and (5.5) can be used to derive the exact and the differential form of the transformation between the associated scalar fields defined in the coset parametrizations (5.4) and (3.5).Furthermore we should also remark that since the exponential factors are not the same in (3.5) and (5.4) by using a similar analysis which is held in the last part of section three one needs to find the local transformations of these different parametrizations.Thus one has to express the coset (5.4) in terms of $\{h_{\widetilde{e}_i}, E_{\widetilde{e}_i\pm e_j}, E_{\widetilde{e}_i\pm\widetilde{e}_j}, E_{\widetilde{e}_i}\}$ by using (5.6) and (5.7) then search for a local transformation law to write it as (3.5).Refering to section four we need only to determine the structure constants of the solvable Lie algebra of $o(10-D, 10-D+N)$ generated by the generators $\{h_{\widetilde{e}_i}, E_{\widetilde{e}_i\pm e_j}, E_{\widetilde{e}_i\pm\widetilde{e}_j}, E_{\widetilde{e}_i}\}$ (the last set $\{E_{\widetilde{e}_i}\}$ is included when N is odd) to find the first-order equations.We see that we do not need to use the $(20-2D+N)$-dimensional fundamental representatives of the generators $\{H^i, E_i^j, V^{ij}, U_I^i\}$ [4] or the ones of $\{h_{\widetilde{e}_i}, E_{\widetilde{e}_i\pm e_j}, E_{\widetilde{e}_i\pm\widetilde{e}_j}, E_{\widetilde{e}_i}\}$ to find the first-order equations.One would certainly need the fundamental representation when one considers the gauge multiplet coupling.Finally the transformation laws would enable us to express the first-order formulations of the scalar cosets constructed in [4] in terms of the original scalar fields which the dimensional reduction contributes.

6 Conclusion

The solution space of the scalar sector of a wide class of supergravity theories is generated by the G/K symmetric space sigma model.After introducing the

algebraic structure and the suitable parametrizations for the coset G/K in section two we have given two equivalent formulations of the non-split coset sigma model and studied them in detail in section three.The solvable Lie algebra parametrization is used to derive the field equations for both of these formulations.In general we have formed an analysis on two different coset maps which define two sets of scalar fields.A local transformation between these two different sets is also constructed in section three.In section four we have generalized the results of [3] for the case when the non-compact symmetry group G is not necessarily in split real form.From the choice of the solvable Lie algebra parametrization given in section two it is apparent that the split case whose dualisation is introduced in [3] for a different spacetime signature convention than the one assumed here is a special example of the general formulation constructed in this note.Thus the formulation given here contains the split case as a limiting example in the group theory sense.We have derived the local first-order equations for a generic non-split coset by using the coset map of [1,2].We have also obtained the first order equations for the scalar fields which are defined through a more conventional coset map by using the explicit transformation defined in section three and by applying a separate dualisation on the later coset map.

In section five we have discussed a possible field of application of our results namely we have presented the link between the abstract formulation of the symmetric space sigma model in section three and four and the scalar coset realizations of [4] which arise in the Kaluza-Klein reduction of the ten dimensional simple supergravity coupled to N Abelian gauge multiplets on T^{10-D}.In particular $N = 16$ case corresponds to the T^{10-D}-compactification of the ten-dimensional low-energy effective heterotic string theory.A transformation between the scalar field definitions given in section three and the original scalar fields used in [4] can be explored by using the transformation of the generators given in section five and by considering the coset parametrizations.This would enable a direct construction of the first-order formulation of the scalar cosets presented in [4].

The dualisation and the local first order formulation of section four is an extension of [3].This work improves the application of the dualisation method of [1,2] from the split-coset case of [3] to the entire set of non-compact symmetric space scalar cosets of supergravities.The results presented here are powerful since they would be the starting point and an essential part of the first order formulations of the pure and the matter coupled supergravities which are not studied in [1,2].

In summary besides studying the field equations and different parametrizations of the non-split coset,this work completes the dual formulation of the symmetric space sigma model when the rigid symmetry group is a real form of a non-compact semisimple Lie group.It extends the construction in [3] which is performed for the special split case.

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